# Accelerator Physics (Ring)

The 1st AOFSRR Summer School - Cheiron School 2007

G. LeBlanc, Australian Synchrotron

# Accelerator Physics (Ring)

- Basics
- Circular Motion
- Dipole
- Quadrupole
- Equations of Motion
- Linear Beam Dynamics
- Matrix Formalism
- Beams and Phase Space
- Betatron Functions
- Beam Optics

# Basics

Equations and Relationships Used to Study the Interactions of Electromagnetic Fields and Charged Particles

# Basics

## Maxwell's Equations

| | |
|---|---|
| $\overline{B}(\overline{r},t)$ | Magnetic Field |
| $\overline{E}(\overline{r},t)$ | Electric Field |
| $\rho(\overline{r},t)$ | Charge Density |
| $\overline{J}(\overline{r},t)$ | Current Density |

# Basic Formalism

## Maxwell's Equations

$$\nabla \times \overline{B} = \mu_0 \overline{J} + \mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$$

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$$

$$\nabla \cdot \overline{B} = 0$$

$$\nabla \cdot \overline{E} = \frac{1}{\varepsilon_0} \rho$$

# Basic Formalism

## Lorentz Force

$$\overline{F} = e\left(\overline{E} + \overline{v} \times \overline{B}\right)$$

- Describes interaction of charged particles with electromagnetic fields

# Basic Formalism

## Lorentz Force

$$\overline{F} = e\left(\overline{E} + \overline{v} \times \overline{B}\right)$$

- Electric Field for Acceleration
  - Force Parallel to Field
- Magnetic Field for Steering
  - Force Perpendicular to Field and Particle Direction
- For a relativistic particle, the force from a 1 Tesla magnetic field corresponds to an electric field of 300 MV/m

# Basic Formalism

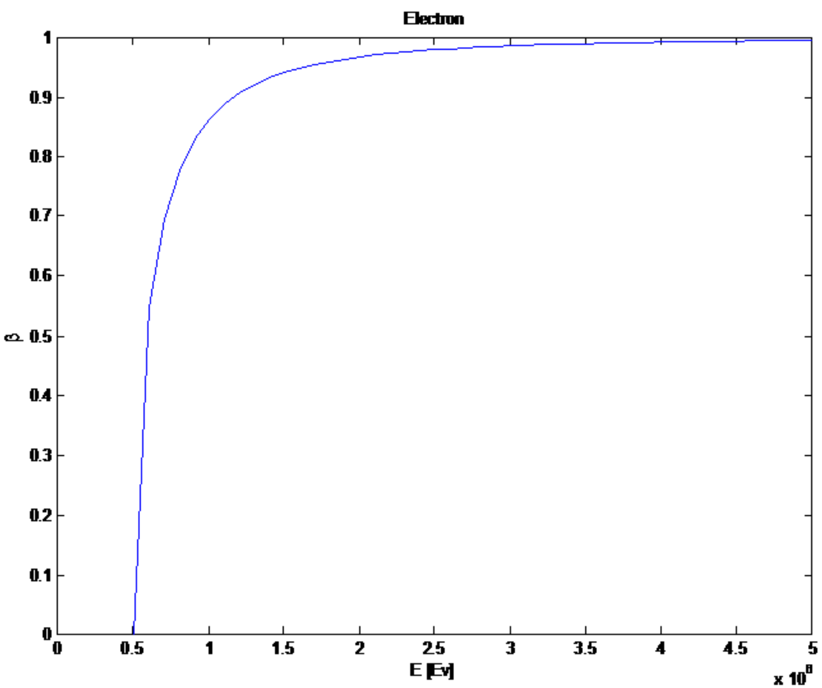
## Energy

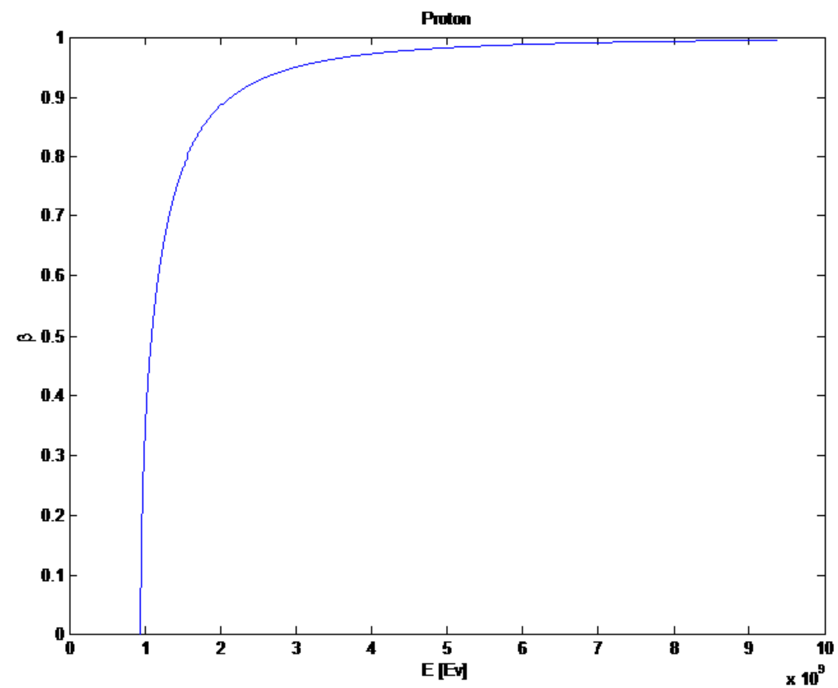
$$E = E_0 + E_{kin}$$

- Rest Energy: $E_0 = m_0 c^2$
- Relativistic Parameter: $\gamma = E / E_0$
- Velocity: $v = \beta c$
- Relativistic Mass: $m = \dfrac{m_0}{\sqrt{1-\beta^2}}$
- Energy in eV: $1eV = 0.16 \cdot 10^{-18} J$

# Basics

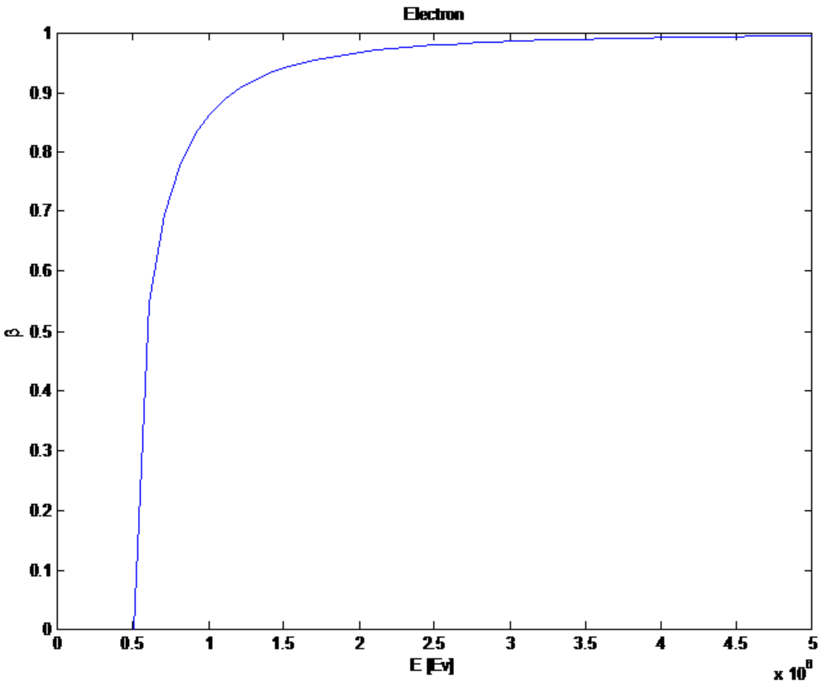
- Particles Relativistic when β≃1

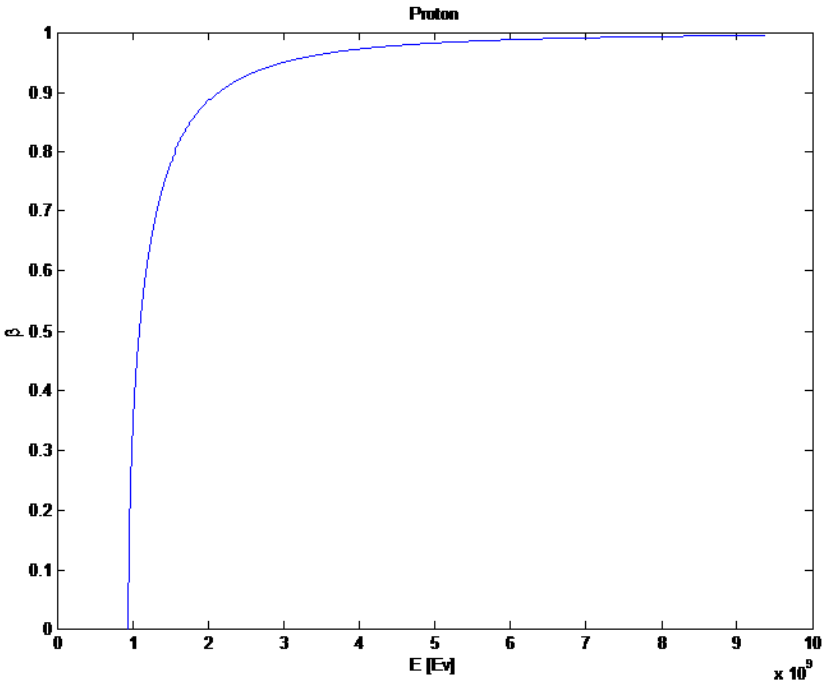


# Basics

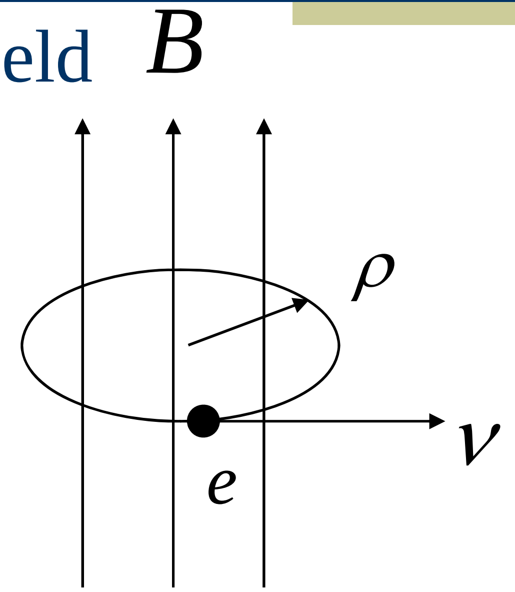
- Particles Relativistic when $\beta \cong 1$



# Circular Motion

- Circular Motion in a Magnetic Field




- Centripetal Force
- Lorentz Force

$$\left.\begin{aligned} F &= \frac{mv^2}{\rho} \\ F &= evB \end{aligned}\right\} \Rightarrow \left\{ \frac{1}{\rho} = \frac{eB}{mv} = \frac{e}{p} B \right.$$

# Dipole

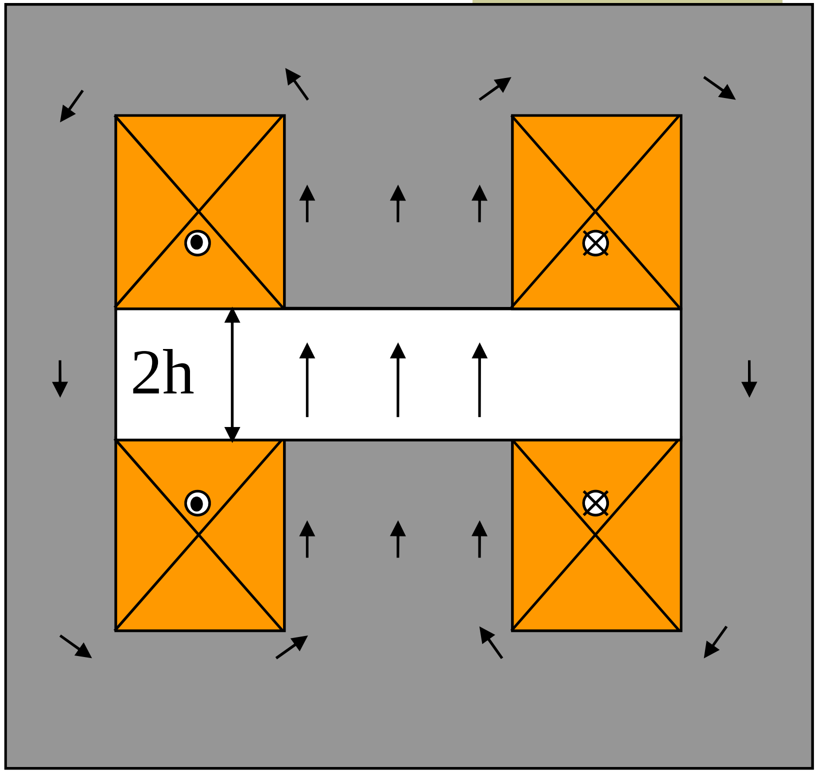
$$B = \frac{nI\mu_0}{h}$$

$$\frac{1}{\rho} = \left| \frac{e}{p} B \right| \Rightarrow \left| B\rho \right| = \frac{p}{e}$$

$$cp = \beta E \Rightarrow$$

$$B\rho[Tm] = 3.33\beta E[GeV]$$

Beam Rigidity



# Quadrupole

## Focusing

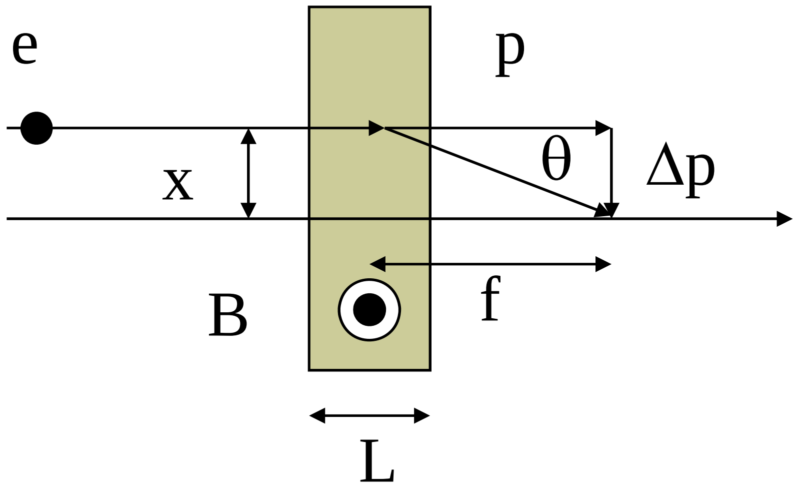
$$\Delta p = F\Delta t = evB_y\left(\frac{L}{v}\right) = eB_yL$$

$$\theta \approx \frac{\Delta p}{p} = \frac{eB_yL}{p}$$

$$B_y = \frac{\partial B_y}{\partial x}x = B'x$$

$$\theta \approx \frac{eB'xL}{p} = \frac{x}{f}$$

$$\frac{1}{f} \approx \frac{eB'L}{p} = \frac{B'L}{(B\rho)}$$

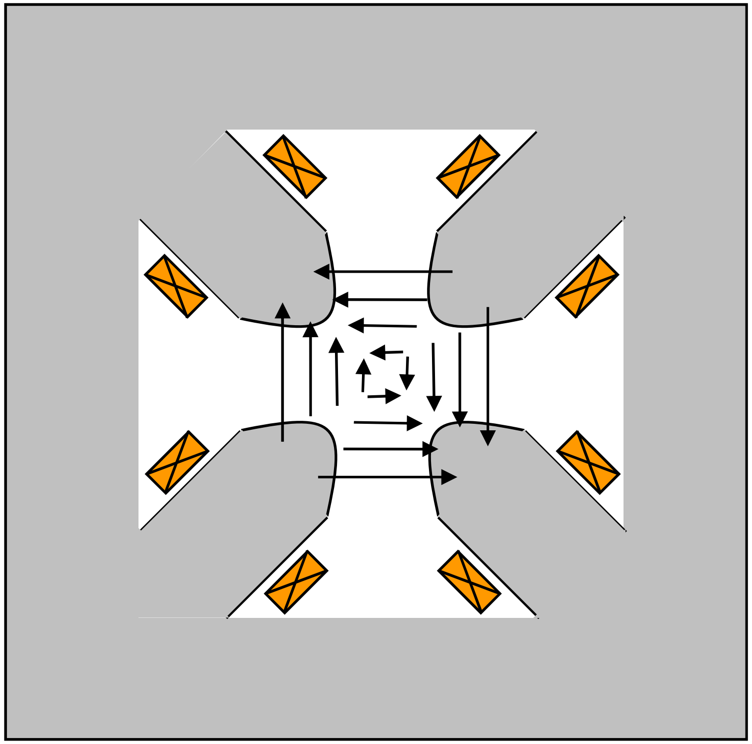


# Quadrupole

- Gradient

$$g = \frac{2\mu_0 nI}{R^2}$$

$$k\left[m^{-2}\right] = 0.3\frac{g\left[T/m\right]}{\beta E\left[GeV\right]}$$



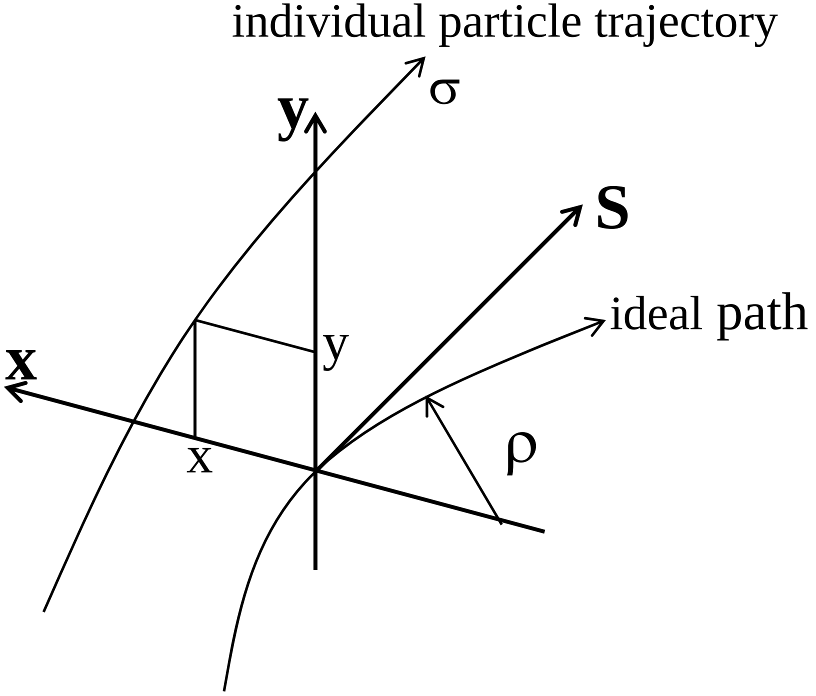
# Equations of Motion

## Coordinate System

- Curvilinear System
- Motion Relative Ideal Path
- Curvature Vector, κ

$$\bar{\kappa} = -\frac{d^2\bar{S}(s)}{ds^2}$$

$$\bar{\kappa} = (\kappa_x, \kappa_y) = (-x'', -y'')$$

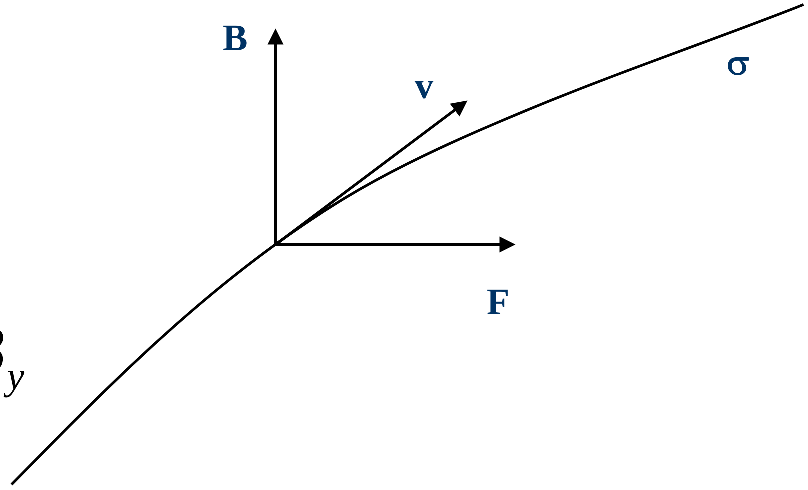


# Equations of Motion

$$\overline{F} = \frac{d\overline{p}}{d\tau} = e\left[\overline{v} \times \overline{B}\right]$$

$$\tau = \sigma / v_\sigma$$

$$\overline{B} = (0, B_y, 0) \Rightarrow \frac{dp_x}{d\tau} = -e v_\sigma B_y$$



# Equations of Motion

Linear Approximation

$$x', y' << 1$$

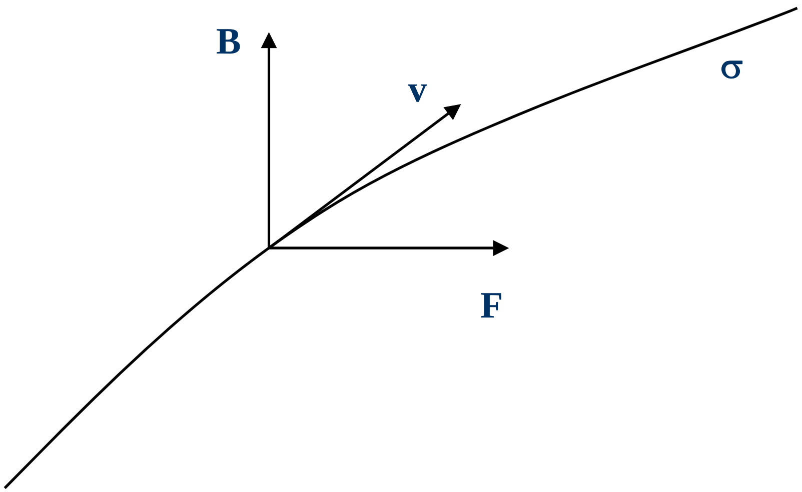
$$p_s = p\sqrt{1 - x'^2 - y'^2} \approx p$$

$$p_x \approx p x'$$

$$x' = \frac{dx}{ds}$$

$$ds \approx d\sigma = v_\sigma d\tau$$

$$\frac{dp_x}{d\tau} \approx p\frac{dx'}{d\tau} \approx v_\sigma p x''$$



# Equations of Motion

$$\frac{dp_x}{d\tau} = -e\,v_\sigma B_y \approx p\,v_\sigma x''$$

$$x'' \approx -\frac{e}{p} B_y$$

$$y'' \approx \frac{e}{p} B_x$$

$$\left| \frac{e}{p} B_{x,y} \right| = \frac{1}{\rho_{x,y}}$$

$$\kappa_x = -x'' \approx \frac{e}{p} B_y$$

$$\kappa_y = -y'' \approx -\frac{e}{p} B_x$$

Local Bending Radii

# Equations of Motion

$$\bar{\kappa}_o = -\frac{d^2\bar{S}_o}{ds^2} \approx -\frac{e}{p}\left[\frac{\bar{v}}{v}\times\bar{B}_o\right]$$

Ideal Path Determined by Bending Magnets

# Linear Beam Dynamics

$$B_x = gy$$

$$B_y = B_{yo} + gx$$

Only Dipole and Quadrupole Fields

$$x'' + \left( \frac{1}{\rho_o^2} + k_o \right) x = 0$$

$$y'' - k_o y = 0$$

Equations of Motion

$$K(s) = \frac{1}{\rho_o^2(s)} + k_o(s)$$

Strength Parameter

# Linear Beam Dynamics

$$u'' + Ku = 0$$

Homogenous Equation

$$K > 0 \Rightarrow \begin{cases} C(s) = \cos\left(\sqrt{K}\, s\right) \\ S(s) = \dfrac{1}{\sqrt{K}} \sin\left(\sqrt{K}\, s\right) \end{cases}$$

$$K < 0 \Rightarrow \begin{cases} C(s) = \cosh\left(\sqrt{|K|} s\right) \\ S(s) = \dfrac{1}{\sqrt{|K|}} \sinh\left(\sqrt{|K|} s\right) \end{cases}$$

Principle solutions

# Linear Beam Dynamics

$$C(0) = 1$$

$$C'(0) = \frac{dC}{ds} = 0$$

$$S(0) = 0$$

$$S'(0) = \frac{dS}{ds} = 1$$

Initial Conditions

# Linear Beam Dynamics

$$u_o$$

$$u'_o$$

Initial Parameters

$$u(s) = C(s)u_o + S(s)u'_o$$

$$u'(s) = C'(s)u_o + S'(s)u'_o$$

Linear Combination of Principle Solutions

# Linear Beam Dynamics

$$u'' + K(s)u = \frac{1}{\rho(s)}\delta$$

Dispersion Treated as Perturbation Term

$$u(s) = aC(s) + bS(s) + \delta D(s)$$
$$u'(s) = aC'(s) + bS'(s) + \delta D'(s)$$

$$\begin{bmatrix} u(s) \\ u'(s) \\ \delta \end{bmatrix} = \begin{bmatrix} C(s) & S(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_o \\ u'_o \\ \delta \end{bmatrix}$$

# Matrix Formalism

$$\begin{bmatrix} x(s) \\ x'(s) \\ y(s) \\ y'(s) \end{bmatrix} = \begin{bmatrix} C_x(s) & S_x(s) & 0 & 0 \\ C'_x(s) & S'_x(s) & 0 & 0 \\ 0 & 0 & C_y(s) & S_y(s) \\ 0 & 0 & C'_y(s) & S'_y(s) \end{bmatrix} \begin{bmatrix} x_o \\ x'_o \\ y_o \\ y'_o \end{bmatrix}$$

# Matrix Formalism

## Drift Space

$$\begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} 1 & s - s_o \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u(s_o) \\ u'(s_o) \end{bmatrix}$$

$$l = s - s_o$$

$$\mathrm{M}_d\left(l|0\right) = \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix}$$

# Matrix Formalism

## Quadrupole Magnet

$$\frac{1}{\rho_o} = 0$$

$$k_o(s) \neq 0$$

$$k_o = |k_o| > 0 \Rightarrow \begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} \cos\psi & \frac{1}{\sqrt{k_o}}\sin\psi \\ -\sqrt{k_o}\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} u(s_o) \\ u'(s_o) \end{bmatrix}$$

Focusing

$$k_o = |k_o| < 0 \Rightarrow \begin{bmatrix} u(s) \\ u'(s) \end{bmatrix} = \begin{bmatrix} \cosh\psi & \frac{1}{\sqrt{|k_o|}}\sinh\psi \\ \sqrt{|k_o|}\sinh\psi & \cosh\psi \end{bmatrix} \begin{bmatrix} u(s_o) \\ u'(s_o) \end{bmatrix}$$

Defocusing

$$\psi = \sqrt{|k_o|}(s - s_o)$$

# Matrix Formalism

## Quadrupole Magnet

$$\varphi = \sqrt{|k_o|}\,l$$

$$\mathrm{M}_{QF}(l|0) = \begin{bmatrix} \cos\varphi & \frac{1}{\sqrt{k_o}}\sin\varphi \\ -\sqrt{k_o}\,\sin\varphi & \cos\varphi \end{bmatrix}$$

$$\mathrm{M}_{QD}(l|0) = \begin{bmatrix} \cosh\varphi & \frac{1}{\sqrt{|k_o|}}\sinh\varphi \\ \sqrt{|k_o|}\,\sinh\varphi & \cosh\varphi \end{bmatrix}$$

# Matrix Formalism

## Thin Lens approximation

$$l << f$$

$$l \to 0$$

$$f^{-1} = k_o l = const.$$

$$\varphi = \sqrt{k_o l} \to 0$$

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} u_o \\ u'_o \end{bmatrix}$$

$$f^{-1} = k_o l > 0$$ Focusing Plane

$$f^{-1} = k_o l < 0$$ Defocusing plane

# Matrix Formalism

## Quadrupole Doublet

$$\mathrm{M} = \mathrm{M}_{Q2}\mathrm{M}_{d}\mathrm{M}_{Q1} =$$

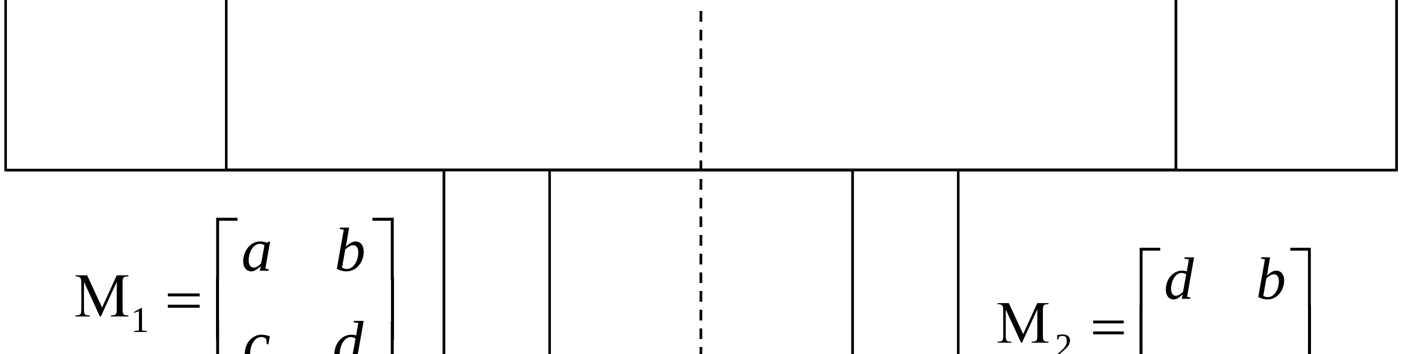
$$\begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}\begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix}\begin{bmatrix} 1-L/f_1 & L \\ -1/f_1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1-L/f_1 & L \\ -1/f^* & 1-L/f_2 \end{bmatrix}$$

$$\frac{1}{f^*} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{L}{f_1 f_2}$$

# Matrix Formalism

## Symmetric Beam Line



$$M_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

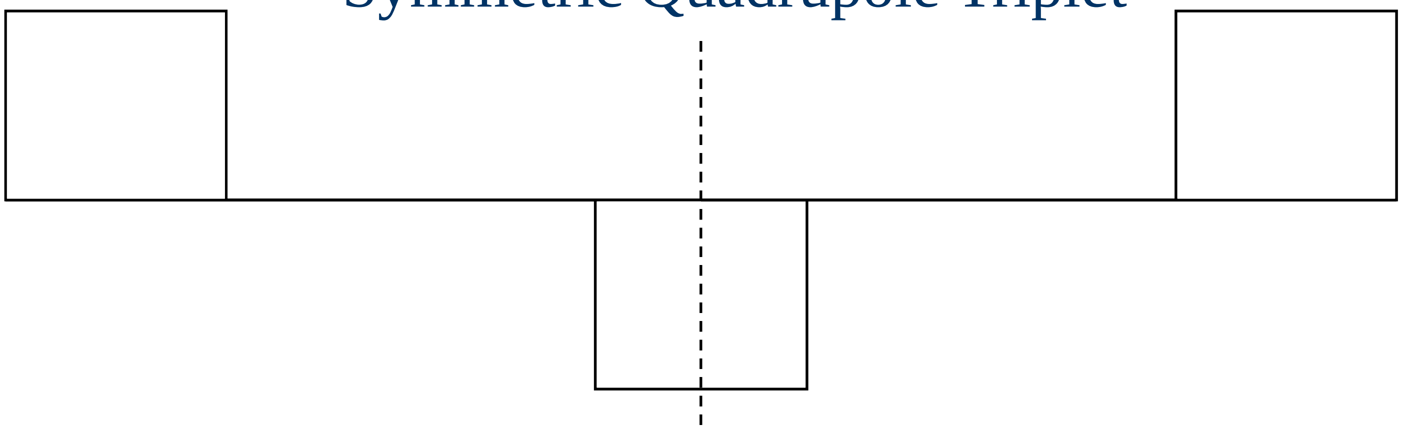
$$M_2 = \begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

$$M_{tot} = \begin{bmatrix} ad + bc & 2bd \\ 2ac & ad + cb \end{bmatrix}$$

$$M_i = \begin{bmatrix} d & -d \\ -c & a \end{bmatrix}$$

# Matrix Formalism

Symmetric Quadrupole Triplet



$$M = \begin{bmatrix} 1 - 2L^2/f^2 & 2L(1 + L/f) \\ -2(1 - L/f)\dfrac{L}{f^2} & 1 - 2L^2/f^2 \end{bmatrix}$$

$$f_1 = f_2 = f$$

FODO channel

Focusing
as long as f>L

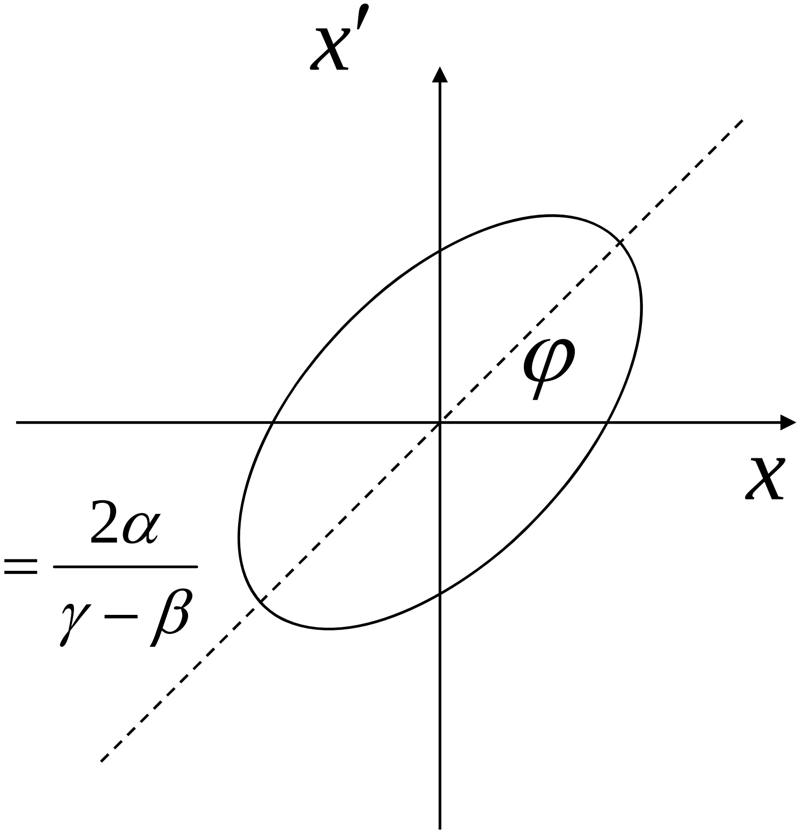
# Beams and Phase Space

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon$$

$$A = \int_{ellipse} dx\,dx' = \pi\varepsilon$$

$$\left.\begin{matrix}\beta \\ \alpha \\ \gamma\end{matrix}\right\} \Rightarrow \beta\gamma - \alpha^2 = 1$$

$$\tan\varphi = \frac{2\alpha}{\gamma - \beta}$$



# Beams and Phase Space

$$\left.\begin{aligned} &\gamma_o x_o^2 + 2\alpha_o x_o x_o' + \beta_o x_o'^2 = \varepsilon \\ &\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{bmatrix} \begin{bmatrix} x_o \\ x_o' \end{bmatrix} \end{aligned}\right\} \Rightarrow$$

$$\left(S'^2\gamma_o - 2S'C'\alpha_o + C'^2\beta_o\right)x^2$$

$$+2\left(-SS'\gamma_o + S'C\alpha_o + SC'\alpha_o - CC'\beta_o\right)xx'$$

$$+\left(S^2\gamma_o - 2SC\alpha_o + C^2\beta_o\right)x'^2 = \varepsilon$$

# Beams and Phase Space

$$\gamma = C'^2\beta_o - 2S'C'\alpha_o + S'^2\gamma_o$$

$$\alpha = -CC'\beta_o + (S'C\alpha_o + SC')\alpha_o - SS'\gamma_o$$

$$\beta = C^2\beta_o - 2SC\alpha_o + S^2\gamma_o$$

# Beams and Phase Space

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} C^2 & -2SC & S^2 \\ -CC' & S'C + SC' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{bmatrix} \begin{bmatrix} \beta_o \\ \alpha_o \\ \gamma_o \end{bmatrix}$$

# Beams and Phase Space

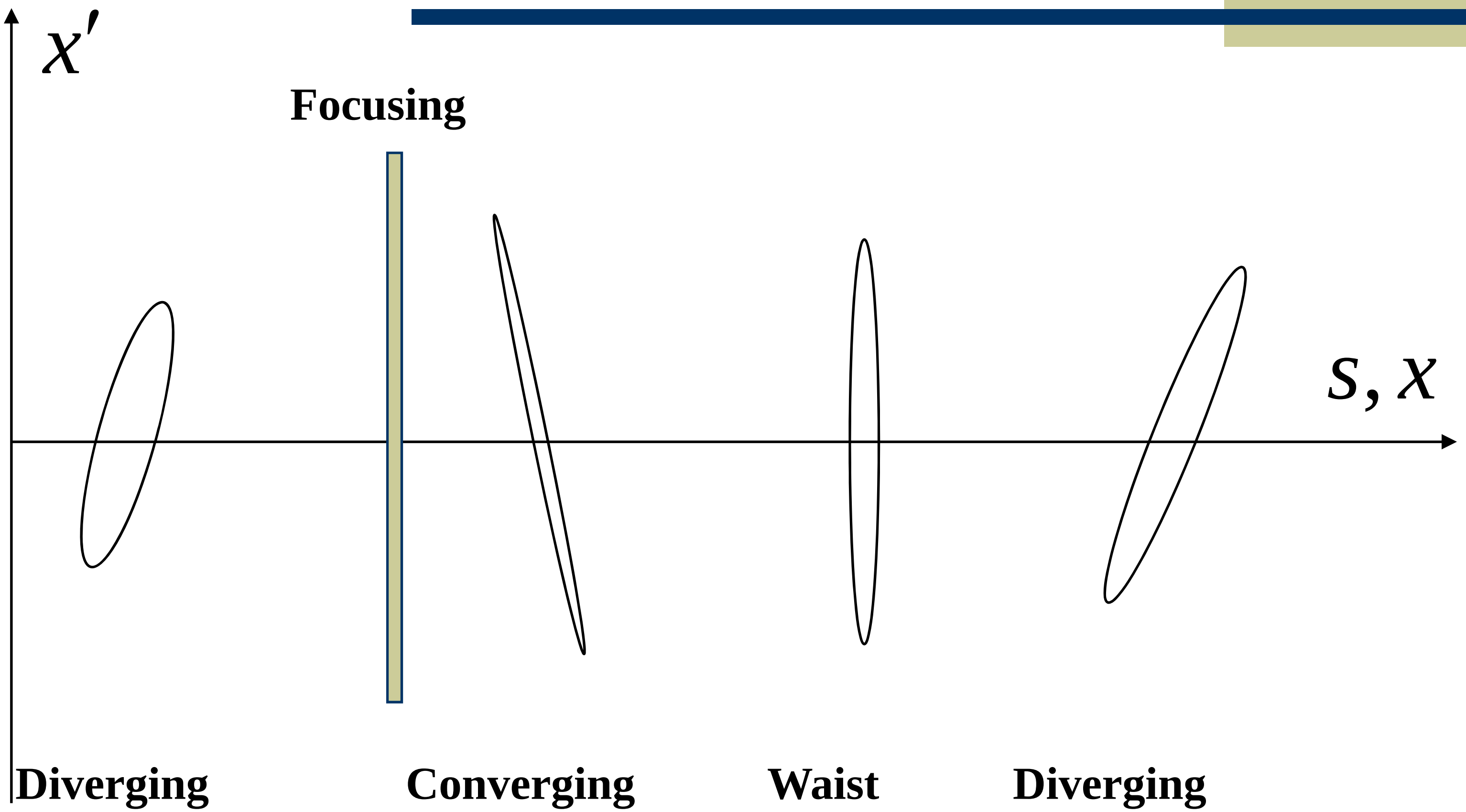
Drift Space

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 & -2l & l^2 \\ 0 & 1 & -l \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_O \\ \alpha_O \\ \gamma_O \end{bmatrix}$$

Beam waist when α=0

$$l = s_w - s_o = \frac{\alpha_o}{\gamma_o}$$

# Beams and Phase Space



# Betatron Functions

$$u(s)=\sqrt{\varepsilon}\sqrt{\beta(s)}\cos[\psi(s)-\psi_o]$$ Ansatz

$$u'=\sqrt{\varepsilon}\,\frac{\beta'}{2\sqrt{\beta}}\cos(\psi-\psi_o)-\sqrt{\varepsilon}\sqrt{\beta}\sin(\psi-\psi_o)\psi'$$

$$u''=\sqrt{\varepsilon}\,\frac{\beta\beta''-1/2\,\beta'^2}{2\beta^{3/2}}\cos(\psi-\psi_o)-\sqrt{\varepsilon}\,\frac{\beta'}{\sqrt{\beta}}\sin(\psi-\psi_o)\psi'$$

$$-\sqrt{\varepsilon}\sqrt{\beta}\sin(\psi-\psi_o)\psi''-\sqrt{\varepsilon}\sqrt{\beta}s\cos(\psi-\psi_o)\psi'^2$$

# Betatron Functions

$$u'' + k(s)u = 0 \Rightarrow$$

$$\frac{1}{2}\left(\beta\beta'' - \frac{1}{2}\beta'^2\right) - \beta^2\psi'^2 + \beta^2 k = 0$$

Sum of sine and cosine terms must vanish

$$\beta'\psi' + \beta\psi'' = 0 \Rightarrow$$

$$\beta\psi' = const = 1$$

$$\psi(s) = \int_0^s \frac{d\bar{s}}{\beta(s)} + \psi_0$$

Phase function

# Betatron Functions

$$\mu(C) = \int_{s}^{s+C} \frac{d\overline{s}}{\beta(s)}$$

Phase Advance

$$\nu_{x,y} = \frac{1}{2\pi} \oint \frac{d\overline{s}}{\beta(s)}$$

Tune

# Betatron Functions

$$\mathrm{M}\left(s+L_p \middle| s\right)=\begin{bmatrix} C & S \\ C' & S' \end{bmatrix}$$

Full lattice period

$$\begin{bmatrix} \beta \\ \alpha \\ \gamma \end{bmatrix}=\begin{bmatrix} C^2 & -2SC & S^2 \\ -CC' & S'C+SC' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{bmatrix}\begin{bmatrix} \alpha_O \\ \beta_O \\ \gamma_O \end{bmatrix}=\mathrm{M}_\beta \cdot \begin{bmatrix} \beta_O \\ \alpha_O \\ \gamma_O \end{bmatrix}$$

# Betatron Functions

General Periodic Solution from Eigenvector Equation

$$\left(\mathrm{M}_{\beta} - \mathrm{I}\right) \cdot \overline{\beta} = 0 \Rightarrow$$

$$\left(C^2 - 1\right)\beta - 2SC\alpha + S^2\gamma = 0$$

$$CC'\beta - \left(S'C + SC' - 1\right)\alpha + SS'\gamma = 0$$

$$C'^2\beta - 2S'C'\alpha + \left(S'^2 = 1\right)\gamma = 0$$

# Betatron Functions

$$\beta^2 = \frac{S^2}{1-C^2}$$

$$\alpha = 0$$

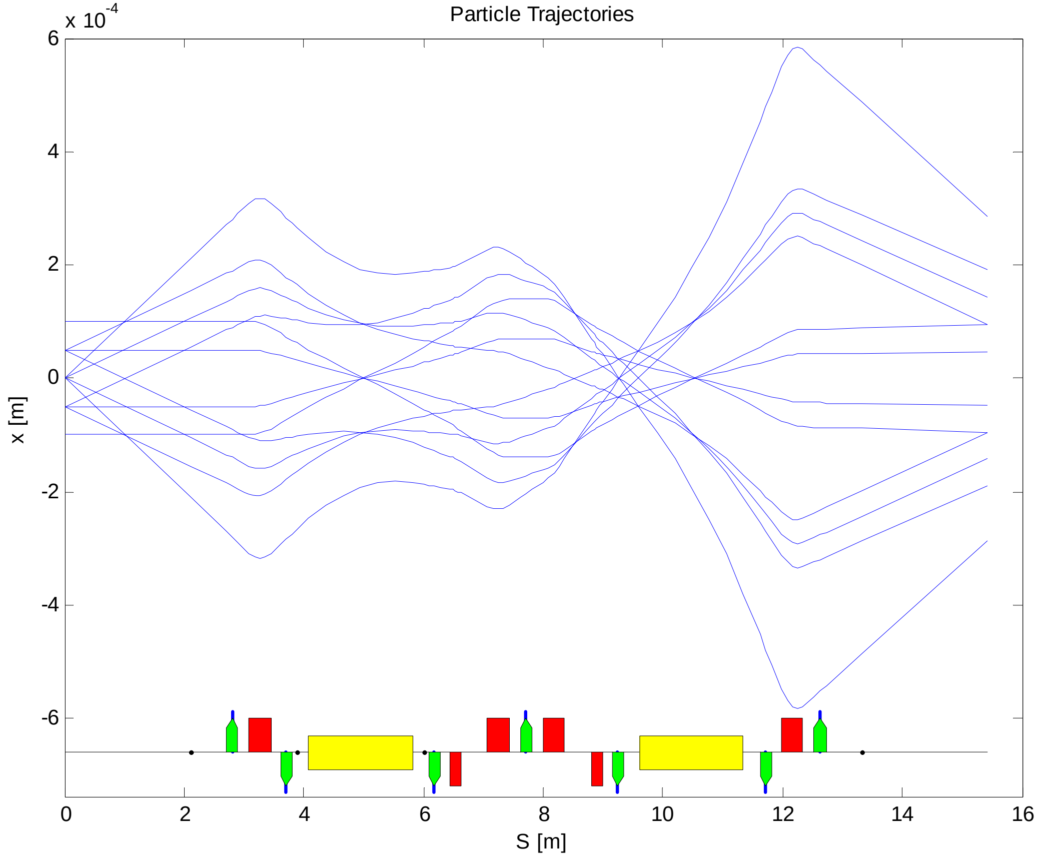
$$\gamma = \frac{1}{\beta}$$

Symmetry Point

$$\mathrm{M} = \begin{bmatrix} \cos\mu & \beta\sin\mu \\ -\frac{1}{\beta}\sin\mu & \cos\mu \end{bmatrix}$$

# Beam Optics

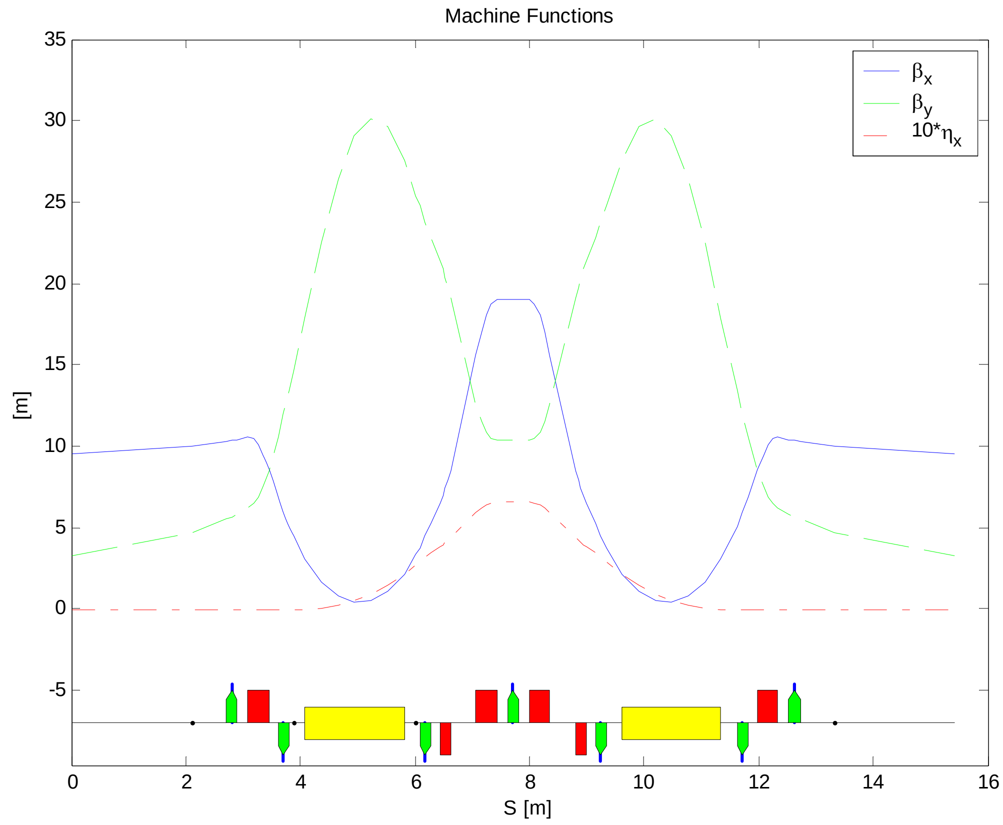
- Particle motion determined by magnetic lattice
- Studied using simulation software




G. LeBlanc Cheiron 2007

# Beam Optics

- Machine Functions
  - Beam Motion
  - Beam Size
  - Beam Emittance



# Beam Optics

- Response Matrix
  - Probe the Machine with the Beam
  - Calibrate Models